Variety and volatility in financial markets

Fabrizio Lillo and Rosario N. Mantegna
Istituto Nazionale per la Fisica della Materia, Unità di Palermo and Dipartimento di Fisica e Tecnologie Relative, Università di Palermo, Viale delle Scienze, I-90128, Palermo, Italy

(Received 4 June 2000)

We study the price dynamics of stocks traded in a financial market by considering the statistical properties of both a single time series and an ensemble of stocks traded simultaneously. We use the *n* stocks traded on the New York Stock Exchange to form a statistical ensemble of daily stock returns. For each trading day of our database, we study the ensemble return distribution. We find that a typical ensemble return distribution exists in most of the trading days with the exception of crash and rally days and of the days following these extreme events. We analyze each ensemble return distribution by extracting its first two central moments. We observe that these moments fluctuate in time and are stochastic processes, themselves. We characterize the statistical properties of ensemble return distribution central moments by investigating their probability density functions and temporal correlation properties. In general, time-averaged and portfolio-averaged price returns have different statistical properties. We infer from these differences information about the relative strength of correlation between stocks and between different trading days. Last, we compare our empirical results with those predicted by the single-index model and we conclude that this simple model cannot explain the statistical properties of the second moment of the ensemble return distribution.

PACS number(s): 05.40.−a, 89.90.+n

I. INTRODUCTION

In recent years physicists have begun to interact with economists to construct models of financial markets [1]. This inspired a group of physicists to analyze and model price dynamics in financial markets using paradigms and tools of statistical and theoretical physics [2]. One goal of this research is to implement a stochastic model of price dynamics in financial markets that reproduces the statistical properties observed in the time evolution of stock prices. Over the past few years, physicists interested in financial analysis have performed several empirical research studies investigating the statistical properties of stock price and volatility time series of a single stock (or of an index) at different temporal horizons [3,4]. This kind of analysis does not take into account any interaction of the considered financial stock with other stocks traded simultaneously on the same market. It is known that the synchronous price returns time series of different stocks are pair correlated [5,6] and several studies have also been performed by physicists to extract information from the correlation properties [7–9]. A precise characterization of collective movements in a financial market is of key importance in understanding the market dynamics and in controlling the associated risk to a portfolio of stocks. The present study contributes to the understanding of collective behavior of a portfolio of stocks in normal and extreme days of market activity.

Specifically, we address the following question: Is the complexity of a financial market essentially limited to the statistical behavior of each financial time series or, rather, does a complexity of the overall market exist? To answer this question, we present the results of an empirical analysis performed adopting the following point of view. We investigate the price returns of an ensemble of *n* stocks simultaneously traded on a financial market on a given day. With this approach, we quantify what we call the *variety* of a financial market on a given trading day [10]. The variety provides statistical information about the amount of varied behavior observed in the stock return in a given ensemble of stocks at a given trading time horizon (in the present case, one trading day). We observe that the distribution of variety is sensitive to the composition of the portfolio investigated (especially to the capitalization of the considered stocks).

The return distribution shows a typical shape for most of the trading days. However, the typical behavior is not observed during crash and rally days. The shape and parameters characterizing the ensemble return distribution are relatively stable during normal phases of the market activity, but become time dependent in the periods following crashes. The variety is characterized by a long-range correlated memory showing that no typical time scale can be expected after a rally or a crash for the expected relaxation to a ''normal'' market phase. Moreover, a simple model such as the single-index model cannot reproduce the statistical properties empirically observed.

The paper is organized as follows. In Sec. II we illustrate our database and the ensemble of stocks considered. Section III is devoted to the investigation of the statistical properties of the time evolution of each single stock. In Sec. IV, we discuss the statistical properties of ensemble return distribution. Specifically, we consider the behavior of the first two central moments, their distribution and time correlation, a comparison of time and portfolio average, and the role of the size and homogeneity of the investigated portfolio. In Sec. V we compare the statistical properties observed in a real financial market with the prediction of the single-index model. In Sec. VI we present a discussion of the results obtained.

II. DATABASE AND VARIABLES INVESTIGATED

The market investigated is the New York Stock Exchange (NYSE) during the 12-year period from January 1987 to December 1998, which corresponds to 3032 trading days. We consider the ensemble of all stocks traded on the NYSE. The number of stocks traded on the NYSE increases in the investigated period, ranging from 1128 at the beginning of 1987 to 2788 at the end of 1998. The total number of data records exceeds six million.

The variable investigated in our analysis is the daily price return, which is defined as

$$R_i(t) \equiv \frac{Y_i(t) - Y_i(t-1)}{Y_i(t-1)}, \tag{1}$$

where $Y_i(t)$ is the closure price of the ith stock at day t ($t = 1,2,\ldots$). For each trading day t, we consider n returns, where n depends on the total number of stocks traded on the NYSE on the selected day t. In our study we use a ''market time.'' With this choice, we consider only the trading days and we remove the weekends and holidays from the calendar time.

A database of more than six million records unavoidably contains some errors. Direct control of a so large database is not realistic. For this reason, to avoid spurious results we filter the data by not considering price returns that are, in absolute values, greater than 50%.

The companies traded on the NYSE are quite different from one another. Differences between the companies are observed both with respect to the sector of their economic interests and with respect to their size. One measure of the size of a company is its capitalization. The capitalization of a stock is the stock price times the number of outstanding shares. In this study, we discuss the role of the differing capitalization in the price dynamics.

III. SINGLE-STOCK PROPERTIES

The distribution of returns with different time horizons of a single stock or index has been studied by several authors [2–6]. The stocks traded on a financial market have different capitalizations. An important issue is whether or not the differences in capitalization are reflected in the statistical properties of the price returns of the stocks. To answer this question, we investigate the distribution of daily returns of 2188 stocks traded on the NYSE.

We compare the statistical properties of daily price return distribution of each stock as a function of its capitalization. We sequence the 2188 stocks in decreasing order according to their capitalization at an arbitrarily chosen day that we select as June 10, 1996. Our sequencing procedure gives to the most capitalized stock (the General Electric Co., GE) the rank $i=1$, to the second one (the Coca Cola Company) the rank $i=2$, and so on. An analysis of the return probability density function (PDF) for the 2188 stocks shows that the distributions are different. This is due in general to (i) different scale and (ii) the different shape of the return PDFs. In order to eliminate one source of difference, we analyze the PDF of the normalized returns $[R_i(t) - \mu_i]/\sigma_i$ ($i = 1,2,\ldots,2188$), where μ_i and σ_i are the first two central moments of the time series $R_i(t)$ defined as

$$\mu_i = \frac{1}{T_i} \sum_{t=1}^{T_i} R_i(t), \tag{2}$$

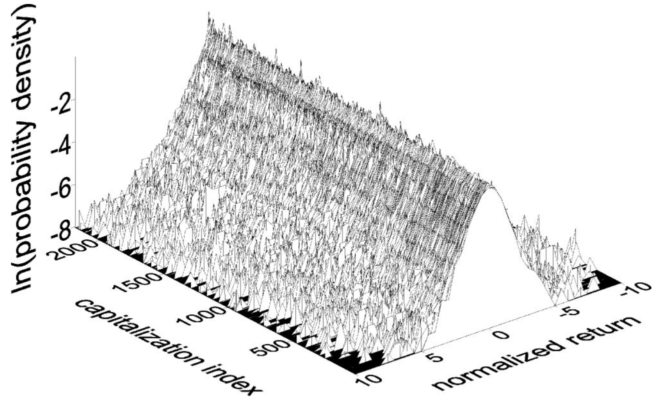

FIG. 1. Surface plot of the logarithm of the probability density function of normalized daily returns $[R_i(t) - \mu_i]/\sigma_i$ of all the stocks traded on the NYSE. The stocks are sorted according to their capitalization on June 10, 1996.

$$\sigma_i = \sqrt{\frac{1}{T_i}\left(\sum_{t=1}^{T_i} (R_i(t) - \mu_i)^2\right)}, \tag{3}$$

where T_i is the number of trading days of the stock i during the investigated period. The quantity μ_i gives a measure of the overall performance of stock i during this period. The standard deviation σ_i is called *historical volatility* in the financial literature and quantifies the risk associated with the ith stock. This quantity is of primary importance in risk management and in option pricing.

The PDF of normalized daily returns of all the stocks sequenced by capitalization is shown in Fig. 1. The central part of the distribution of the most capitalized stocks has a bell-shaped profile. Moving towards less capitalized stocks, the central part of the distribution becomes more peaked and the tails of the distribution become fatter. The PDF of the less capitalized stocks is therefore more leptokurtic than the PDF of the more capitalized ones.

The typical estimation of the degree of leptokurtosis of a PDF is done by considering its kurtosis. The evaluation of the kurtosis of the PDF is, in general, difficult for a small set of data because the fourth moment and all the moments higher than the second are extremely sensitive to the highest absolute returns. This implies that the kurtosis calculated from a relatively small set of records is dominated by the highest absolute returns rather than by the shape of the PDF and therefore it is not a good statistical estimate. To avoid this problem, we quantify the distance between the empirically calculated PDF of the daily returns of the ith stock and the Gaussian distribution by considering the quantity

$$h \equiv \frac{\langle |x| \rangle}{\sqrt{\langle x^2 \rangle - \langle x \rangle^2}}. \tag{4}$$

The quantity h is nondimensional and depends on the first two moments. For the Gaussian distribution

$$P_G(x) = \frac{1}{\sqrt{2\pi\sigma_G^2}} \exp\left(-\frac{(x-\mu_G)^2}{2\sigma_G^2}\right), \tag{5}$$

the parameter h is equal to

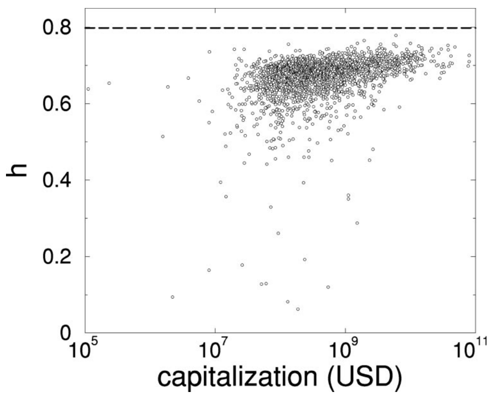

FIG. 2. Each circle represents the h parameter defined in Eq. (4) of the daily return distribution of a stock as a function of its capitalization. The dashed line is the value $\sqrt{2/\pi}\simeq 0.80$, which is the lower bound for h_G expected for a Gaussian distribution of daily return. Values of h smaller than h_G indicate a leptokurtic distribution of returns. The parameter h slowly increases by increasing the capitalization.

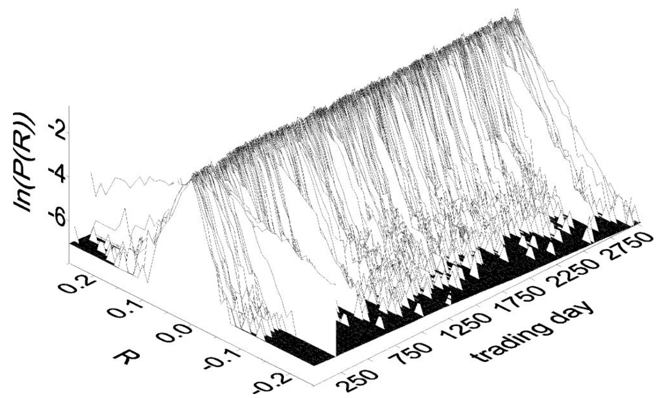

FIG. 3. Surface plot of the logarithm of the ensemble return distribution for the 12-year period of investigation from January 1987 to December 1998. In this figure the 1987 crash is clearly recognizable (trading day index equal to 200) and the high-volatility two-year period 1997–1998 (trading day index from 2500 to 3032).

$$h_G=\sqrt{\frac{2}{\pi}}\left[\exp\left(-\frac{\mu_G^2}{2\sigma_G^2}\right)+\sqrt{\frac{\pi}{2}}\frac{\mu_G}{\sigma_G}\mathrm{Erf}\left(\frac{\mu_G}{\sqrt{2}\sigma_G}\right)\right]. \quad (6)$$

The parameter h_G is a function of the ratio μ_G/σ_G ranging from the lower bound $\sqrt{2/\pi}$ when $\mu_G/\sigma_G=0$ to infinity. For a leptokurtic PDF, as, for example, a Laplace distribution or a Student's t distribution with finite variance, h is always smaller than h_G. The distance of h from h_G quantifies the degree of leptokurtosis of the PDF considered. Figure 2 shows the parameter h for the stocks traded on the NYSE as a function of their capitalization. In this figure, we also show the lower bound of h_G for comparison. The empirically calculated parameter h is systematically smaller than h_G. The mean value $\langle h\rangle$ of the overall market is $\langle h\rangle=0.67$ and its standard deviation is $\sigma_h=0.06$. Hence this result suggests that, as a first approximation, one can assume that the large majority of stocks are characterized by a roughly similar PDF. However, we wish to point out that this conclusion is only valid as a first approximation because a trend of h is clearly detected in Fig. 2. Specifically, h increases as the capitalization increases. Therefore, the less capitalized stocks have a more leptokurtic daily return PDF than the more capitalized ones.

The second moment of return distribution has been found finite in recent research [11–14]. In order to verify the convergence of the PDF towards a Gaussian PDF at large temporal horizons, we evaluate the h parameter for weekly $\langle h_w\rangle$ and monthly $\langle h_m\rangle$ return PDFs. We obtain from our analysis $\langle h_w\rangle=0.70$ and $\langle h_m\rangle=0.74$. These results show that the values of h moves towards $h_G=\sqrt{2/\pi}\simeq 0.80$ when the time horizon of returns is increased, supporting the conclusion of finite second moment.

IV. ENSEMBLE RETURN DISTRIBUTION

In the preceding section we focused on the statistical properties of the time evolution of price returns for each single stock traded on the NYSE. In this section we perform a synchronous analysis on the return of all stocks traded on the NYSE. Toward this end we extract the n returns of the n stocks for each trading day t. The distribution of these returns $P_t(R)$ provides information about the kind of activity occurring in the market on the selected trading day t.

Figure 3 shows the logarithm of the PDF as a function of the return and of the trading day. In this figure we show the interval of daily returns from -25% to 25%. The central part of the distribution is roughly triangular in a logarithmic scale and this shape and its scale are conserved for long time periods. Sometimes the shape and scale of the ensemble return PDF changes abruptly in the presence of either large average positive returns or large average negative returns. Figure 4 shows the same data as Fig. 3 in a contour plot. The contour lines describe equiprobability regions. In order to point out the properties of the central part of the distribution, in Fig. 4 we plot only the returns that are less than 15% in absolute value. Only a few points of the contour lines lay outside this limit during the 1987 and 1998 crises. In Fig. 4 there are long time periods in which the central part of the distribution maintains its shape and the equiprobability contour lines are approximately parallel one to each other. As an example, one can consider the three-year period 1993-1995. On the other hand, there are time periods in which the shape of the distribution changes drastically. In general, these periods corresponds to financial turmoil in the market. For example a dramatic change in the shape and scale of the PDF is observed in Fig. 4 during and after the Oct. 19, 1987 crash, at the beginning of 1991, and at the end of 1998. A systematic analysis of the change of the shape and scale of the ensemble return distribution during extreme events of the market has been discussed elsewhere [15]

One key aspect of the ensemble return distribution concerns its shape during the normal periods of activity of the market. Is the distribution approximately Gaussian or is a systematic deviation from a Gaussian shape quantitatively observed? We already cited that a direct inspection of Fig. 3 suggests that the central part of the empirical return distribution is roughly Laplacian (triangular in a logarithmic scale) and not Gaussian. To make this analysis more quantitative, we show in Fig. 5 the ratio between the value of h determined for each trading day from the ensemble return distri-

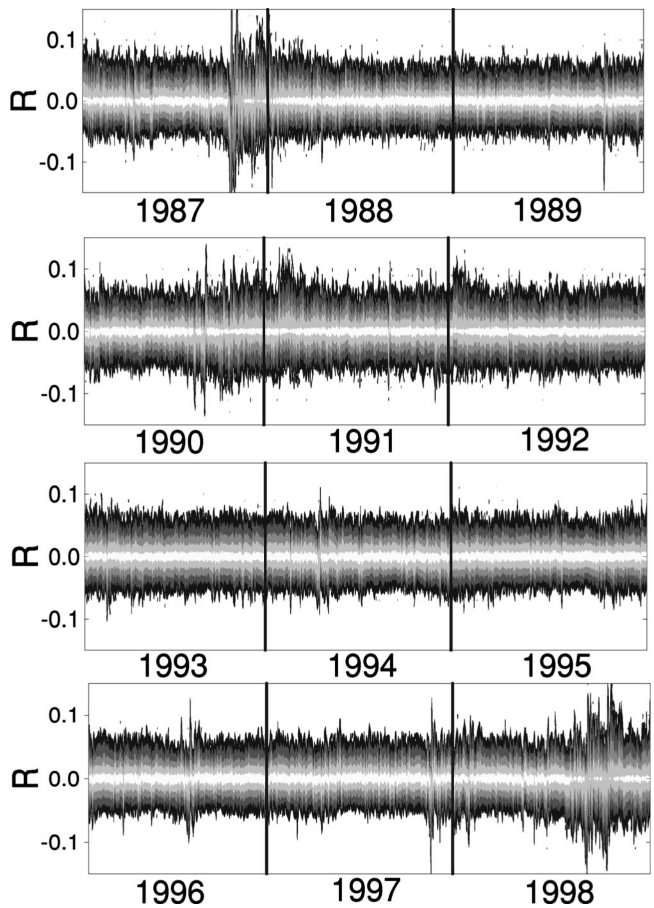

FIG. 4. Contour plot of the logarithm of the ensemble return distribution for the 12-year period January 1987 to December 1998 (same data as in Fig. 3). The contour plot is obtained for equidistant intervals of the logarithmic probability density. The brightest area of the contour plot corresponds to the most probable value.

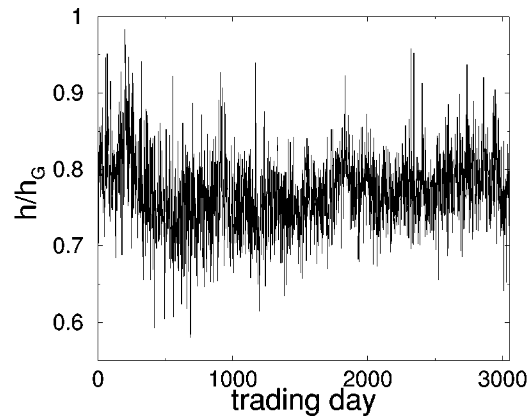

FIG. 5. Ratio between the h parameter defined in Eq. (4) of the ensemble return distribution and the value of h_G expected from a Gaussian distribution and defined by Eq. (6) for each trading day. The ratio h/h_G is systematically smaller than one, indicating that the ensemble return distribution is leptokurtic for each trading day.

bution and the quantities h_G calculated by determining the mean and the standard deviation of $P_t(R)$ and hypothesizing a Gaussian shape by using Eq. (6). The ratio h/h_G is systematically smaller than one and this implies that the Gaussian hypothesis for the shape of the distribution is not verified by the empirical analysis. In other words, the Gaussian distribution is not a good approximation for either the central part or the tails of the distribution, and the deviation from the Gaussian behavior is systematically observed for all trading days of the 12-year time period analyzed in our study.

In summary, the ensemble return distribution well characterizes the market activity. It has a typical shape and scale during long periods of ''normal'' activity of the market characterized by moderately low average daily returns. During extreme events the shape and scale change dramatically in a systematic way. Specifically, during crises the ensemble return distribution becomes negatively skewed, whereas during rallies a positive skewness is observed [15]. Figure 4 clearly shows that extreme events (such as the October 1987 crash) trigger an ''aftershock'' period in the ensemble return PDF that can last for a period of several months.

A. Central moments

In order to characterize more quantitatively the ensemble return distribution on day t, we extract the first two central moments on each of the 3032 trading days. Specifically, we consider the average and the standard deviation defined as

$$\mu(t)=\frac{1}{n_t}\sum_{i=1}^{n_t} R_i(t), \tag{7}$$

$$\sigma(t)=\sqrt{\frac{1}{n_t}\left(\sum_{i=1}^{n_t}[R_i(t)-\mu(t)]^2\right)}, \tag{8}$$

where n_t indicates the number of stocks traded on day t.

The mean of price returns $\mu(t)$ quantifies the general trend of the market at day t. The standard deviation $\sigma(t)$ provides a measure of the width of the ensemble return distribution. We call this quantity the *variety* of the ensemble because it provides a measure of the variety of behavior observed in a financial market on a given day. A large value of $\sigma(t)$ indicates that different companies are characterized by rather different returns on day t. In fact, on days of high variety some companies experience great gains whereas others have great losses. The mean and the standard deviation of price returns are not constant, but fluctuate in time. We study the temporal series of $\mu(t)$ and $\sigma(t)$ in order to characterize the temporal evolution of the ensemble return distribution quantitatively. We investigate these fluctuating parameters by investigating their time correlation properties and their PDFs.

B. Probability distributions of the central moments

The empirical PDF of the mean $\mu(t)$ for the 3032 trading days investigated is shown in Fig. 6. The central part of this distribution is non-Gaussian and is roughly described by a Laplace distribution.

The mean $\mu(t)$ is proportional to the sum of n random variables $R_i(t)$ $(i=1,2,\ldots,n)$. The central limit-theorem prescribes that the sum of n *independent* random variables with finite variance converges to a Gaussian PDF. By assuming a finite value for the volatility of stocks, the observation that the PDF of the mean return $\mu(t)$ is non-Gaussian can be therefore attributed to the presence of correlation between the stocks.

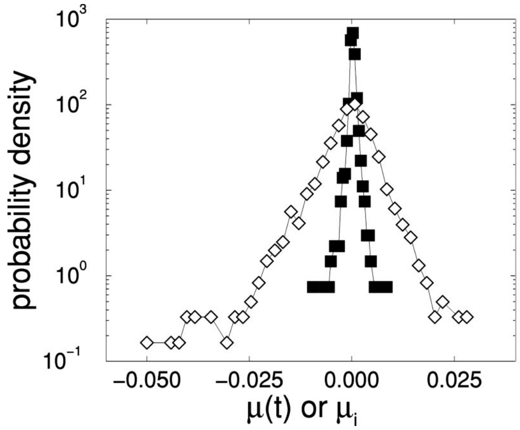

FIG. 6. Linear-log plot of the probability density function of the mean $\mu(t)$ of the ensemble return distribution (white diamonds) and of the mean of the daily return μ_i of all the stocks traded in the NYSE (black squares).

Figure 7 shows the PDF of the variety $\sigma(t)$. The central part of this distribution is approximated by a log-normal distribution. A deviation from the log-normal behavior is observed in the tail of higher values of variety. This deviation depends on the size of the portfolio and will be discussed in subsection IV E.

C. Correlations in the central moments

Another important statistical property of $\mu(t)$ and $\sigma(t)$ concerns their correlation properties [10]. For the considered portfolio, we calculate the autocorrelation function of a variable $x(t)$ which is defined as

$$R(\tau)\equiv\frac{\langle x(t)x(t+\tau)\rangle-\langle x(t)\rangle\langle x(t+\tau)\rangle}{\langle x(t)^2\rangle-\langle x(t)\rangle^2}. \tag{9}$$

In agreement with previous results [10], we find that the mean $\mu(t)$ is approximately delta correlated, whereas the autocorrelation function of $\sigma(t)$ is long-range correlated. The empirical autocorrelation function of $\sigma(t)$ is well approximated by a power-law function $R(\tau)\propto\tau^{-\delta}$. By performing a best fit with a maximum time lag of 50 trading days, we determine the exponent $\delta=0.230\pm0.006$. This result indicates that the variety $\sigma(t)$ has a long-range memory in the market. We recall that the historical volatility is characterized by long-range memory of the same nature [16–18].

Another way to investigate the long-range correlation is to determine the power spectrum of the investigated variable. We evaluate the power spectrum of $\sigma(t)$ and we perform a best fit of the power spectrum with a functional form of the kind

$$S(f)\propto\frac{1}{f^{\eta}}. \tag{10}$$

Our best fit for the power spectrum of $\sigma(t)$ gives for the exponent $\eta\approx1.1$. This result confirms that the variety $\sigma(t)$ is a long-range time-correlated random variable.

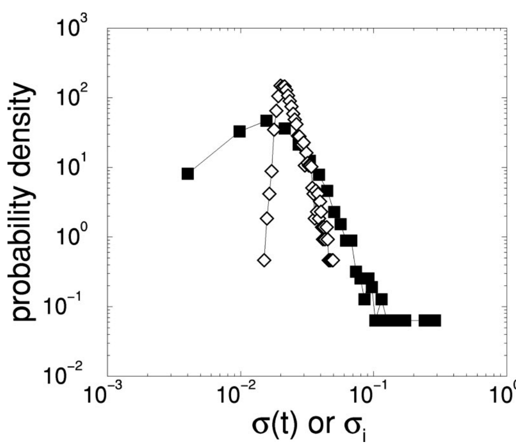

FIG. 7. Log-log plot of the probability density function of the variety $\sigma(t)$, i.e., the variance of the ensemble return distribution (white diamonds) and of the volatility σ_i, i.e., the variance of the daily return, of the all the stocks traded in the NYSE (black squares).

D. Time and portfolio average

Figure 6 shows two curves. In fact, in Fig. 6 we also show the PDF of the mean μ_i. The quantity μ_i [see Eq. (2)] is the mean return of stock i averaged over the investigated time interval. The PDF of μ_i is non-Gaussian and it is much more peaked than the PDF of $\mu(t)$. Hence, the statistical behavior observed by investigating a large portfolio in a market day is not representative of the statistical behavior observed by investigating the time evolution of single stocks.

This comparison can be performed as well for the second moment of the distributions. In Fig. 7 we compare the PDF of the volatility σ_i with the PDF of the variety $\sigma(t)$. Also in this case, the statistical properties of σ_i and $\sigma(t)$ are different. Specifically, the PDF of $\sigma(t)$ is more peaked than the PDF of σ_i.

In order to understand the differing behavior of the time-averaged and the portfolio-averaged quantities, for the sake of simplicity we consider a portfolio composed by N stocks traded in a period of T trading days. We first study the properties of the two means, μ_i and $\mu(t)$. It is straightforward to verify that

$$\langle\mu_i\rangle_i=\langle\mu(t)\rangle_t\equiv\mu, \tag{11}$$

where $\langle\cdots\rangle_t$ indicates temporal average and $\langle\cdots\rangle_i$ indicates ensemble average. The variances of μ_i and $\mu(t)$ are, in general, different. We obtain for the variance of $\mu(t)$ the expression

$$\mathrm{Var}[\mu(t)]\equiv\frac{1}{T}\sum_{t=1}^{T}[\mu(t)-\mu]^2=\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_{ij}^2, \tag{12}$$

where σ_{ij}^2 is the return covariance between stocks i and j defined as

$$\sigma_{ij}^2=\langle R_i(t)R_j(t)\rangle_t-\langle R_i(t)\rangle_t\langle R_j(t)\rangle_t. \tag{13}$$

The width of the PDF of $\mu(t)$ (shown in Fig. 6) is the square root of $\mathrm{Var}[\mu(t)]$. Equations (12) and (13) indicate that this quantity depends on both the ensemble-averaged square volatility [terms with $i=j$ in Eq. (12)] and the mean of the synchronous cross-covariances between pairs of stocks [terms with $i\neq j$ in Eq. (12)].

With similar methods we show that the variance of μ_i can be written as

$$\mathrm{Var}[\mu_i]\equiv\frac{1}{N}\sum_{i=1}^{N}(\mu_i-\mu)^2=\frac{1}{T^2}\sum_{t=1}^{T}\sum_{t'=1}^{T}\sigma_{tt'}^2, \quad (14)$$

where we define the return covariance between trading days t and t' as

$$\sigma_{tt'}^2=\langle R_i(t)R_i(t')\rangle_i-\langle R_i(t)\rangle_i\langle R_i(t')\rangle_i. \quad (15)$$

This quantity gives an estimate of the correlation present in the whole portfolio on trading days t and t'. The double sum in Eq. (14) can be split into a term depending on the average square variety ($t=t'$) and a term depending on the correlation between different trading days ($t\neq t'$).

We verify that the average square variance and volatility satisfy the sum rule

$$\mathrm{Var}[\mu_i]+\langle\sigma_i^2\rangle_i=\mathrm{Var}[\mu(t)]+\langle\sigma^2(t)\rangle_t. \quad (16)$$

Combining Eqs. (12), (14), and (16) we show that

$$\frac{T-1}{T}\langle\sigma^2(t)\rangle_t+\frac{2}{N^2}\sum_{j=1}^{N}\sum_{i<j}\sigma_{ij}^2=\frac{N-1}{N}\langle\sigma_i^2\rangle_i+\frac{2}{T^2}\sum_{t=1}^{T}\sum_{t'<t}\sigma_{tt'}^2. \quad (17)$$

Since $N,T\gg 1$, we approximate $(N-1)/N\cong(T-1)/T\cong 1$ and Eq. (17) becomes

$$\langle\sigma_i^2\rangle_i-\langle\sigma^2(t)\rangle_t\cong\langle\sigma_{ij}^2\rangle_{i\neq j}-\langle\sigma_{tt'}^2\rangle_{t\neq t'} \quad (18)$$

or, equivalently,

$$\mathrm{Var}[\mu(t)]-\mathrm{Var}[\mu_i]\cong\langle\sigma_{ij}^2\rangle_{i\neq j}-\langle\sigma_{tt'}^2\rangle_{t\neq t'}. \quad (19)$$

Figure 6 shows that $\mathrm{Var}[\mu(t)]>\mathrm{Var}[\mu_i]$. This empirical observation, together with the last relation, tells us that the synchronous cross-correlations between the stocks are, on average, stronger than the single-stock correlation present in the whole portfolio on two different trading days. This result is consistent with previous observations that synchronous returns of different stocks are significantly cross-correlated [5–9], whereas single price returns are poorly autocorrelated in time. This conclusion is also verified by our empirical observation that $\langle\sigma_i^2\rangle_i>\langle\sigma^2(t)\rangle_t$.

E. Portfolio size

One key aspect of the previous results concerns the degree of generality of the observed stylized facts. In other words, are the empirical properties of the variety dependent upon the portfolio considered? In Sec. II we have shown that all the stocks are not equivalent with respect to their statistical properties (see the spread of points observed in Fig. 2).

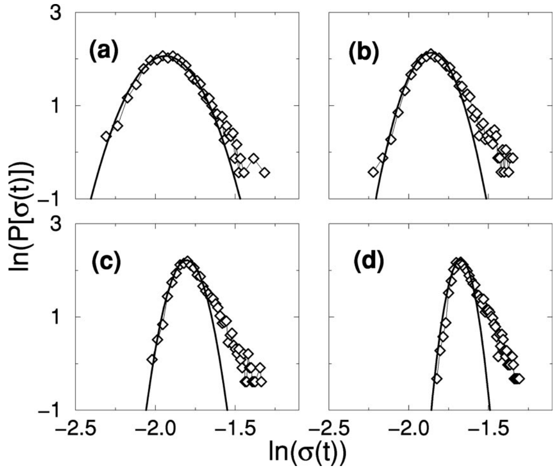

FIG. 8. Log-log plot of the probability density function of the variety $\sigma(t)$ for the four considered ensemble of stocks. (a) DJIA30, (b) SP100, (c) SP500, (d) NYSE. The solid lines are our best fit of the central part of the distribution according to a lognormal distribution.

In fact, a trend is observed in the degree of non-Gaussian shape of the return distribution as a function of the stock capitalization.

To test the degree of sensitivity of our results to the average capitalization of the selected portfolio, we repeat the analysis presented in subsection III B for three other portfolios of stocks traded on the NYSE. Specifically, we investigate (a) the set of 30 stocks used to compute the Dow Jones Industrial Average index, (b) the set of stocks traded on the NYSE and used to compute the Standard & Poor's 100 index, and (c) the set of stocks traded on the NYSE and used to compute the Standard & Poor's 500 index. The results obtained for all the stocks traded on the NYSE are also considered for reference. The four sets are different with respect to two aspects. They differ in the number of stocks present in the set and in the average capitalization of the stocks considered. The empirical PDFs of $\mu(t)$ for the four sets considered are roughly the same. An evident different behavior is observed for the variety. In Fig. 8 we show the PDF of the variety of the portfolios of stocks considered. Specifically, panels (a), (b), (c), and (d) of Fig. 8 are the results obtained for the Dow Jones 30, Standard & Poor's 100, Standard & Poor's 500, and NYSE sets of stocks, respectively. By moving from the smallest to the largest stock portfolio, two effects take place. The PDF of the variety becomes progressively sharper and deviates more from a log-normal profile. The fact that the PDF of the variety becomes progressively sharper is probably due to the fact that the number of elements in the set considered increases, whereas we interpret the progressive deviation from the log-normal profile as a direct manifestation of the progressive increase of the degree of inhomogeneity of the stock portfolio.

In summary, the presence of inhomogeneity in capitalization in the stock portfolio affects the statistical properties of the variety, of the portfolio. This fact should be kept in mind when results about the variety such as results about other statistical properties including return distribution, are obtained by considering the statistical properties of a set of inhomogeneous stocks.

V. SINGLE-INDEX MODEL

In this section we compare the results of our empirical analysis obtained for the NYSE portfolio of stocks with the results obtained by modeling the stock price dynamics with the single-index model. The single-index model [5,6] is a basic model of price dynamics in financial markets. It assumes that the returns of all stocks are controlled by one factor, usually called the ''market.'' In this model, for any stock i we have

$$R_i(t)=\alpha_i+\beta_i R_M(t)+\epsilon_i(t), \qquad (20)$$

where $R_i(t)$ and $R_M(t)$ are the return of the stock i and of the ''market'' on day t, respectively, α_i and β_i are two real parameters, and $\epsilon_i(t)$ is a zero mean noise term characterized by a variance equal to $\sigma^2_{\epsilon_i}$. The noise terms of different stocks are assumed to be uncorrelated, $\langle \epsilon_i(t)\epsilon_j(t)\rangle_t=0$ for $i\neq j$. Moreover, the covariance between $R_M(t)$ and $\epsilon_i(t)$ is set to zero for any i.

Each stock is correlated with the ''market'' and the presence of such a correlation induces a correlation between any pair of stocks. It is customary to adopt a broad-based stock index for the ''market'' $R_M(t)$. Our choice for the market time series is the Standard and Poor's 500 index. The best estimate of the model parameters α_i, β_i, and $\sigma^2_{\epsilon_i}$ is done with the ordinary least-squares method [6]. In order to compare our empirical results with those predicted by the single-index model, we build up an artificial market according to Eq. (20). To this end, we first evaluate the model parameters for all the stocks traded on the NYSE and then we generate a set of n surrogate time series according to Eq. (20) by using the customary assumption of Gaussian statistics for ϵ_i. To make the simulation as realistic as possible, in the generation of our surrogate data set we use as ''market'' time series the true time series of the Standard and Poor's 500 index.

We evaluate the central moments $\mu(t)$ and $\sigma(t)$ defined in Eqs. (7),(8) for the surrogate data. In Fig. 9(a) we show the time series of $\mu(t)$ of the real data and in Fig. 9(b) we show the same quantity for the surrogate market data generated according to the single-index model. The agreement between the two time series is pretty high and therefore the single-index model describes quite well the mean returns of the market at time t provided that the behavior of the ''market'' $R_M(t)$ is known. This result is also confirmed by Fig. 10, where the PDFs of $\mu(t)$ for real and surrogate data are shown. Also, the time correlation properties of surrogate $\mu(t)$ are pretty similar to the real ones. In fact, a fast decaying autocorrelation function of $\mu(t)$ is observed in surrogate data. Good agreement is also observed when one investigates the statistical properties of μ_i and σ_i. The single-index model approximates quite well the empirical distribution of μ_i and σ_i.

Different behavior is observed for the variety $\sigma(t)$. Figures 9(c) and 9(d) show the time series of $\sigma(t)$ for real and surrogate data, respectively. The real time series of the variety is nonstationary and shows several bursts of activity. Conversely, the surrogate time series is quite stationary with the exception of the 1987 crash.

One important point is to consider if these results are still observed when non-Gaussian statistics is assumed for the random variables ϵ_i. Non-Gaussian statistics is indeed observed in the empirical analysis of stock returns [2–6,12–14]. For example, a generalized single-index model with non-Gaussian noise terms has been shown to describe some of the statistical properties of stock returns observed in a real market [19]. To test if our results are still valid for a single-index model with non-Gaussian noise terms, we generate, as in Ref. [19], a second set of n surrogate time series, still according to Eq. (20) but assuming that $\epsilon_i=\sigma_{\epsilon_i}w$, where w is a random variable distributed according to Student's t density function

$$P(w)=\frac{C_\kappa}{(1+w^2/\kappa)^{(\kappa+1)/2}}, \qquad (21)$$

where C_κ is a normalization constant. Empirical investigations of real data [13,14,20] suggest a value between 4 and 6

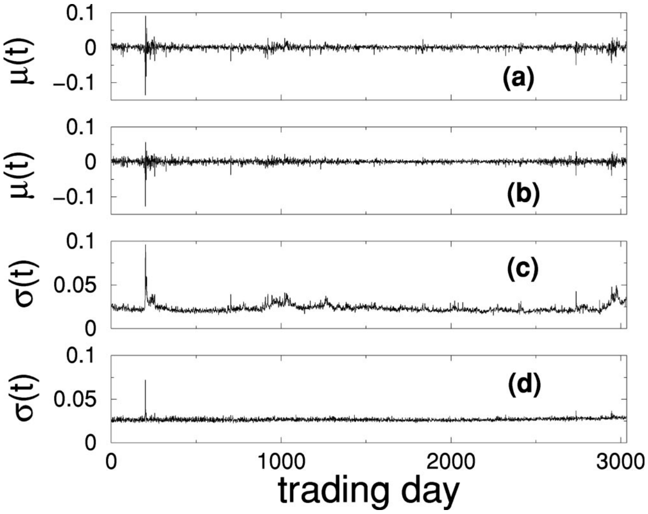

FIG. 9. (a) Time series of the mean of the ensemble return distribution $\mu(t)$. (b) Time series of the mean of the ensemble return distribution for the surrogate data generated according to the single-index model. (c) Time series of the variety $\sigma(t)$ of the ensemble return distribution. (d) Time series of the variety of the ensemble return distribution for the surrogate data generated according to the single-index model.

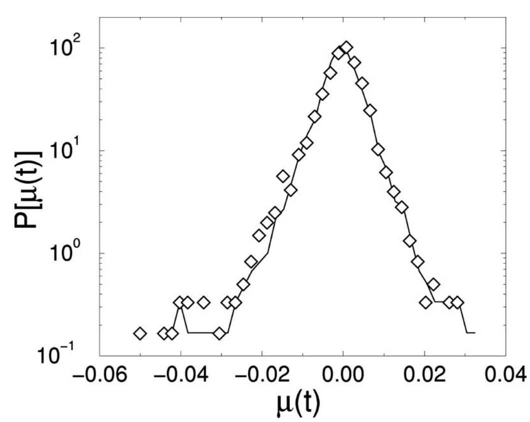

FIG. 10. Comparison of the probability density function of the mean $\mu(t)$ of the ensemble return distribution obtained from real data (diamonds) with that obtained from surrogate data generated according to the single-index model (continuous line).

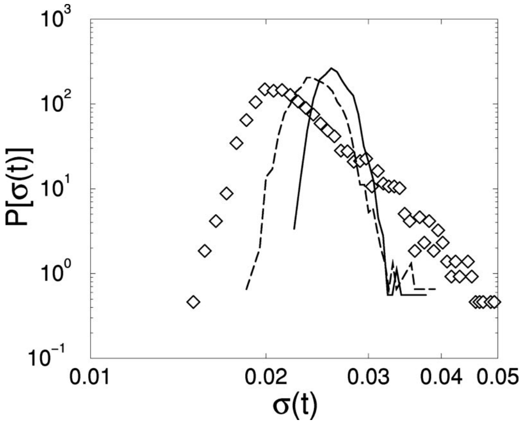

FIG. 11. Comparison of the probability density function of the variety $\sigma(t)$ obtained from real data (diamonds) with that obtained from surrogate data generated according to the single-index model with Gaussian noise terms (continuous line) or with student's t noise terms with probability density of Eq. (21) with $\kappa=3$ (dashed line).

for the power-law exponent of $P(w)$ for large values of $|w|$. In our simulation we take the most leptokurtic distribution within this interval that corresponds to $\kappa=3$. We also verify that for greater values of κ the behavior of the surrogate data is intermediate between that of the single-index model with customary Gaussian statistics and that of the single-index model with Student's t statistics and $\kappa=3$.

By repeating the previous investigation for the new series of data, we evaluate the central moments $\mu(t)$ and $\sigma(t)$. As in the Gaussian case, we find good agreement between the statistical properties of $\mu(t)$ of real and surrogate data. Figure 11 shows the PDF of $\sigma(t)$ for real and surrogate data for both Gaussian and non-Gaussian noise terms. Both the single-index model with Gaussian noise terms and the one with Student's t noise terms fail to describe the real distribution of $\sigma(t)$.

In summary, the single-index model gives a good approximation of the statistical behavior of $\mu(t)$, μ_i and σ_i, whereas it describes poorly the statistical behavior of the variety of a portfolio of stocks traded in a financial market. This conclusion is also supported by the observation that the autocorrelation function of the variety decays to the value 0.1 in $2-3$ trading days both under the assumption of Gaussian and Student's t noise terms. On the other hand, long-range correlation of $\sigma(t)$ is observed in real data.

A more refined analysis shows that the artificial ensemble return distribution obtained with Gaussian noise terms is systematically less leptokurtic than the real one, whereas the artificial ensemble return distribution obtained with Student's t noise terms better mimics the properties of leptokurtosis of the real ensemble return distribution. Moreover, in Ref. [15] we show that the single-index model is unable to predict the change in the symmetry properties of the ensemble return distribution on crash and rally days. The differences observed between the behavior of real data and the behavior of surrogate data suggest that the correlations among the stocks can be explained by the single-index model only for normal periods in first approximations, whereas the model completely fails to reproduce the correlation behavior during extreme events.

VI. CONCLUSIONS

The present study shows that one needs to consider not only the statistical properties characterizing the time evolution of price for each stock traded but also the synchronous collective behavior of the portfolio considered to reveal the overall complexity of a financial market. We show that such collective behavior of a stock portfolio is efficiently monitored by the variety of the ensemble return distribution. This variable is directly observable for each portfolio and presents interesting statistical properties. It is non-Gaussian distributed and long-range correlated. The detailed statistical properties depend on the considered portfolio. We verify that for a portfolio characterized by comparable capitalization, the distribution of the variety is approximately log-normal. Deviations from the log-normal behavior are observed for less homogeneous (in capitalization) portfolios.

The shape of the distribution and the long-term memory of the variety are not reproduced by considering surrogate data simulated by using a single-index model with a realistic time series for the "market." This implies that the complexity detected by the empirical analysis performed cannot be modeled with a similar simple stock price model. The correlations present in the market are more complex than those hypothesized by the single-index model.

The correct modeling of the statistical properties of the variety can then be used as a benchmark for stock price models more sophisticated than the single-index model.

The ensemble return distribution shows a qualitatively and quantitatively different behavior in normal and extreme trading days. The variety of a portfolio is then able to detect quite clearly shocks and aftershocks occurring in the market. Hence, it is a promising direct observable capable of measuring how much pressure a portfolio is under and how distant it is from typical market activity on a specific trading day. A theoretical challenge would be to relate this empirical ensemble observation directly to the correlations active between pairs of stocks of a portfolio.

In summary, we believe that the overall complexity of a financial market can be detected and modeled only by considering simultaneously (i) the statistical properties of the time evolution of stock prices of the portfolio considered *and* (ii) the statics and dynamics of the correlations existing between stocks.

ACKNOWLEDGMENTS

The authors thank INFM and MURST for financial support. This work constitutes part of the FRA-INFM project *Volatility in financial markets*. F.L. acknowledges FSE-INFM for financial support. We wish to thank Giovanni Bonanno for help with numerical calculations.

[1] *The Economy as an Evolving Complex System* edited by P. W. Anderson, K. J. Arrow, and D. Pines (Addison-Wesley, Redwood City, CA, 1988).
[2] For a collection of papers, see, for example, *Econophysics: Proceedings of the Budapest Workshop*, edited by J. Kertesz and I. Kondor (Kluwer, Dordrecht, in press); Proceedings of the International Workshop on Econophysics and Statistical Finance, edited by R.N. Mantegna [Physica A **269** (1999)]; Proceedings of the EPS Conference on Applications of Physics in Financial Analysis, edited by J.-P. Bouchaud [Int. J. Theor. Appl. Finance (to be published)].
[3] R. N. Mantegna and H. E. Stanley, *An Introduction to Econophysics: Correlations and Complexity in Finance*, (Cambridge University Press, Cambridge, 2000).
[4] J.-P. Bouchaud and M. Potters, *Theory of Financial Risk*, (Cambridge University Press, Cambridge, England, 2000).
[5] E. J. Elton and M. J. Gruber, *Modern Portfolio Theory and Investment Analysis* (Wiley & Sons, New York, 1995).
[6] J. Y. Campbell, A. W. Lo, and A. C. MacKinlay, *The Econometrics of Financial Markets* (Princeton University Press, Princeton, NJ, 1997).
[7] R.N. Mantegna, Eur. Phys. J. B **11**, 193 (1999).
[8] L. Laloux, P. Cizeau, J.-P. Bouchaud, and M. Potters, Phys. Rev. Lett. **83**, 1467 (1999).
[9] V. Plerou, P. Gopikrishnan, B. Rosenow, L.A.N. Amaral, and H.E. Stanley, Phys. Rev. Lett. **83**, 1471 (1999).
[10] F. Lillo and R. N. Mantegna, Int. J. Theor. Appl. Finance (to be published); e-print cond-mat/9909302.
[11] V. Akgiray, and G.G. Booth, J. Business Econ. Stat. **6**, 51 (1988).
[12] R.N. Mantegna and H.E. Stanley, Nature (London) **376**, 46 (1995).
[13] T. Lux, Appl. Financial Econ. **6**, 463 (1996).
[14] P. Gopikrishnan, M. Meyer, L.A.N. Amaral, and H.E. Stanley, Eur. Phys. J. B **3**, 139 (1998).
[15] F. Lillo and R.N. Mantegna, Eur. Phys. J. B **15**, 603 (2000).
[16] M.M. Dacorogna, U.A. Müller, R.J. Nagler, R.B. Olsen, and O.V. Pictet, J. Intl. Money Finance **12**, 413 (1993).
[17] Y. Liu, P. Gopikrishnan, P. Cizeau, M. Meyer, C.-K. Peng, and H.E. Stanley, Phys. Rev. E **60**, 1390 (1999).
[18] M. Pasquini and M. Serva, Physica A **269**, 140 (1999); Economics Lett. **65**, 275 (1999).
[19] P. Cizeau, M. Potters, and J.-P. Bouchaud, e-print cond-mat/006034.
[20] V. Plerou, P. Gopikrishnan, L.A.N. Amaral, M. Meyer, and H.E. Stanley, Phys. Rev. E **60**, 6519 (1999).